

BRANCHED COVERS OF ELLIPTIC CURVES AND KÄHLER GROUPS WITH EXOTIC FINITENESS PROPERTIES

CLAUDIO LLOSA ISENRIK

ABSTRACT. We construct Kähler groups with arbitrary finiteness properties by mapping products of closed Riemann surfaces holomorphically onto an elliptic curve: for each $r \geq 3$, we obtain large classes of Kähler groups that have classifying spaces with finite $(r-1)$ -skeleton but do not have classifying spaces with finitely many r -cells. We describe invariants which distinguish many of these groups. Our construction is inspired by examples of Dimca, Papadima and Suciu.

1. INTRODUCTION

A *Kähler group* is a group which can be realized as fundamental group of a compact Kähler manifold. In particular, every Kähler group is finitely presented. A group G is of *finiteness type* \mathcal{F}_r if there is a $K(G, 1)$ with finite r -skeleton; so type \mathcal{F}_1 is equivalent to finite generation and type \mathcal{F}_2 is equivalent to finite presentability.

In 2009, Dimca, Papadima and Suciu [7] constructed the first examples of Kähler groups that are of finiteness type \mathcal{F}_{r-1} , but not of type \mathcal{F}_r , for all $r \geq 3$. In fact their groups are *projective*, that is, fundamental groups of smooth projective manifolds. In particular, these groups, although torsion-free, can not have compact Kähler manifolds as classifying spaces. Our main goal here is to construct broader classes of such groups.

We will show how to construct large classes of new examples of Kähler groups with arbitrary finiteness properties from surjective maps of a direct product of surface groups onto an elliptic curve. More precisely, we will prove

Theorem 1.1. *Let $r \geq 3$ and let $g_1, \dots, g_r \geq 2$. Let E be an elliptic curve and, for $i = 1, \dots, r$, let S_{g_i} be a closed connected surface of genus g_i together with a branched covering $f_{g_i} : S_{g_i} \rightarrow E$. Assume that the map*

$$f = \sum_{i=1}^r f_{g_i} : S_{g_1} \times \cdots \times S_{g_r} \rightarrow E$$

induces a surjective map on fundamental groups.

Then the generic fibre H of f is connected and f induces a short exact sequence

$$1 \rightarrow \pi_1 H \rightarrow \pi_1 S_{g_1} \times \cdots \times S_{g_r} \rightarrow \pi_1 E \rightarrow 1$$

on fundamental groups. Moreover, the group $\pi_1 H$ is a projective (and thus Kähler) group of type \mathcal{F}_{r-1} , but not of type \mathcal{F}_r .

Under the additional assumption that all maps f_{g_i} are *purely branched*, we will give the following complete classification of all Kähler groups that can be obtained using our construction.

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The notion of a purely branched covering will be defined in Definition 2.2. Roughly speaking a purely branched covering is a branched covering which is obtained from the base space by alone branching.

Theorem 1.2. *Let E be an elliptic curve. Let $r, s \geq 3$, let $g_i, h_j \geq 2$ and let S_{g_i} be a closed Riemann surface of genus $g_i \geq 2$ and let R_{h_i} be a closed Riemann surface of genus $h_i \geq 2$ for $1 \leq i \leq r$ and $1 \leq j \leq s$. Assume that there are purely branched k_i -fold holomorphic covering maps $p_i : S_{g_i} \rightarrow E$ and purely branched l_i -fold holomorphic covering maps $q_i : R_{h_i} \rightarrow E$. Define $p = \sum_{i=1}^r p_i : S_{g_1} \times \cdots \times S_{g_r} \rightarrow E$ and $q = \sum_{j=1}^s q_j : R_{h_1} \times \cdots \times R_{h_s} \rightarrow E$ and denote by H_p and H_q the smooth generic fibres of p , respectively q .*

Then the Kähler groups $\pi_1 H_p$ and $\pi_1 H_q$ are isomorphic if and only if $r = s$ and there is a permutation of the R_{h_i} and q_i such that $g_i = h_i$ and $k_i = l_i$ for $i = 1, \dots, r$.

The construction is inspired by the work of Dimca, Papadima and Suciu, but is more general. The construction given by Dimca, Papadima and Suciu is a topological construction and does not lead to an explicit finite presentation. In [9], Llosa Isenrich derived an explicit presentation for these examples, answering a question of Suciu. The original proof that Dimca, Papadima and Suciu's examples have arbitrary finiteness properties consists of an involved argument making use of characteristic varieties. Alternative proofs have been given since by Biswas, Mj and Pancholi [2] and by Suciu [10]. We will give a shorter proof making use of the classification of the finiteness properties of subgroups of direct products of surface groups obtained by Bridson, Howie Miller and Short [3].

Biswas, Mj and Pancholi [2] suggested a more general approach for arbitrary irrational Lefschetz pencils over surfaces of positive genus with singularities of Morse type. Although it is not explicit in [2], the class of examples constructed in our work can also be obtained as a consequence of their work. However, the techniques are quite different: Their approach is based on topological Lefschetz fibrations which by definition have Morse type singularities and the result for non-Morse type singularities then follows by a deformation argument. They do not develop any techniques to distinguish between different examples (cf. Theorem 1.2 above).

As an example of our construction we will produce a class of Kähler groups which to us seem like the most natural analogue in the Kähler setting of a specific subclass of the Bestvina-Brady groups [1]. Using our classification we will then show that these groups are not isomorphic to Dimca, Papadima and Suciu's groups. An interesting question that one can ask here is if one can generalise our construction further to produce natural Kähler analogues for all of the original Bestvina Brady groups. Notice that Dimca, Papadima and Suciu showed that the only original Bestvina Brady groups which are Kähler are free abelian groups of even rank [6, Corollary 1.3].

The structure of the work is as follows: In Section 2 we prove that the fibres of our maps are connected. In Section 3 we will prove an algebraic result which will allow us to deduce the finiteness properties of the groups arising from our construction. In Section 4 we explain our general construction which we then apply in Section 5 to realise the algebraically described groups from Section 3 as Kähler groups. In Section 6 we explain how the question about the existence of isomorphisms between Kähler groups obtained using our construction reduces to a question in linear algebra. In Section 7 we solve this question for maps arising from purely

branched coverings and conclude that the groups in Section 5 are not isomorphic to Dimca, Papadima and Suciu's groups.

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2. CONNECTEDNESS OF FIBRES

In this section we prove that the fibres of the maps in Theorem 1.1 are connected.

Theorem 2.1. *Let $r \geq 2$, let E be an elliptic curve and let S_{g_i} be a closed Riemann surface of genus $g_i \geq 2$. Let $f_{g_i} : S_{g_i} \rightarrow E$ be holomorphic branched covers of E .*

The map $f = \sum_{i=1}^r f_{g_i} : S_{g_1} \times \cdots \times S_{g_r} \rightarrow E$ has connected fibres if and only if it induces a surjective map on fundamental groups.

Proof. Every holomorphic map $h : X \rightarrow Y$ between compact complex manifolds X and Y with connected fibres induces a surjective map on fundamental groups, since it is a locally trivial fibration over the complement of a complex codimension one subvariety of Y . Hence, if f has connected fibres then it induces a surjective map on fundamental groups.

Assume now that f induces a surjective map on fundamental groups. If f does not have connected fibres then Stein factorisation yields a closed Riemann surface S and holomorphic maps $\alpha : S \rightarrow E$ and $\beta : S_{g_1} \times \cdots \times S_{g_r} \rightarrow S$ such that α is finite-to-one and β has connected fibres. Since holomorphic finite-to-one maps between closed Riemann surfaces are branched covering maps it follows that α is a branched covering.

Choose a base point $(p_1, \dots, p_r) \in S_{g_1} \times \cdots \times S_{g_r}$ and denote by β_i the restriction of β to the i th factor $\{(p_1, \dots, p_{i-1})\} \times S_{g_i} \times \{(p_{i+1}, \dots, p_r)\}$. Then there is $e_i \in E$ such that $\alpha \circ \beta_i = e_i + f_{g_i}$. Since β_i is holomorphic it is non-trivial and finite-to-one and thus also a finite-sheeted holomorphic branched covering map. It follows that $\beta_{i\#}(\pi_1 S_{g_i}) \leq \pi_1 S$ is a finite index subgroup and therefore not cyclic for $i = 1, \dots, r$.

It then follows from [2, Lemma 7.1] that S must itself be an elliptic curve. Since the argument is short we want to give it here: Let $\gamma_1 \in \pi_1 S_{g_1}$ and $\gamma_2 \in \pi_1 S_{g_2}$ such that their images $\beta_1 \circ \gamma_1$ and $\beta_2 \circ \gamma_2$ do not lie in a common cyclic subgroup of $\pi_1 S$. Then $\beta_1 \circ \gamma_1$ and $\beta_2 \circ \gamma_2$ generate a \mathbb{Z}^2 -subgroup of $\pi_1 S$ and the only closed Riemann surfaces with \mathbb{Z}^2 -subgroups are elliptic curves. Thus, S is an elliptic curve.

Using Euler characteristic it follows that any branched covering map between 2-dimensional tori is an unramified covering map. By assumption the map $f = \alpha \circ \beta$ induces a surjective map on fundamental groups. Hence, the map $\alpha : S \rightarrow E$ is an unramified holomorphic covering which is surjective on fundamental groups and therefore $S = E$ and α is biholomorphic. In particular, the map f has connected fibres. \square

We introduce a special class of branched covering maps of tori. Let X be a closed connected manifold and let Y be a torus of the same dimension, let $D_1 \subset Y$ be a complex analytic subvariety

of codimension at least one or more generally a real analytic subvariety of codimension at least two. Let $f : X \rightarrow Y$ be a branched covering map with branching locus D_1 , that is, $f^{-1}(D_1)$ is a nowhere dense set in X mapping onto D_1 .

Assume that there are points $p_1, \dots, p_l \in D_1$ and $g_1, \dots, g_k \in \pi_1(Y \setminus D_1)$ such that

$$\pi_1(Y \setminus D_1) = \langle \mu_1, \dots, \mu_k, b_1, \dots, b_l \rangle$$

and for any choice $U_1, \dots, U_l \subset Y$ of open neighbourhoods of p_1, \dots, p_l there are paths $\delta_1, \dots, \delta_l : [0, 1] \rightarrow Y \setminus D_1$ starting at a fixed base point $z^0 \in Y \setminus D_1$ and ending in U_i and loops $\nu_i : [0, 1] \rightarrow U_i \setminus D_1$ such that the concatenation $\beta_i = \delta_i \cdot \nu_i \cdot \delta_i^{-1}$ is a representative of b_i based at z^0 .

Definition 2.2. We call the map f *purely branched* if it satisfies the condition

$$\langle \langle \mu_1, \dots, \mu_k \rangle \rangle \leq f_{1*}(\pi_1(X_1 \setminus f_1^{-1}(D_1))).$$

Note that for a Y a 2-torus, X a closed connected surface of genus $g \geq 1$ and $f : X \rightarrow Y$ a branched covering map with branching locus b_1, \dots, b_r , the map Y is purely branched if there is a choice of representatives $\mu_1, \mu_2 : [0, 1] \rightarrow Y \setminus D_1$ of generators of $\pi_1 Y$ such that every lift of μ_1, μ_2 is a loop in X .

3. A KÄHLER ANALOGUE OF BESTVINA BRADY GROUPS

A large classes of groups with prescribed finiteness type are the Bestvina Brady groups which were constructed using combinatorial Morse theory [1]. These groups arise as kernels of surjective maps from Right Angled Artin groups to the integers.

Given a finite simplicial graph Γ , denote by $V(\Gamma)$ its vertex set and by $E(\Gamma)$ its edge set. We define the *Right Angled Artin group* (RAAG) A_Γ for Γ as the group with the finite presentation

$$A_\Gamma = \langle V(\Gamma) \mid [v, w] \text{ if } vw \in E(\Gamma) \rangle.$$

Associated to every such presentation of a RAAG is a natural homomorphism

$$\begin{aligned} \phi_\Gamma : \quad A_\Gamma &\rightarrow \mathbb{Z} = \langle t \rangle \\ V(\Gamma) \ni v &\mapsto t \end{aligned}$$

Associated to the RAAG A_Γ we have the flag complex X_Γ with 1-skeleton Γ , that is, the simplicial complex obtained from Γ by requiring that whenever a finite set of vertices in Γ is pairwise connected, there is a simplex in X_Γ with these vertices.

The *Bestvina Brady group* BB_Γ associated to Γ is defined as $BB_\Gamma = \ker \phi_\Gamma$. The finiteness properties of the Bestvina Brady groups are completely understood:

Theorem 3.1 (Bestvina Brady Theorem, [1]). *Let Γ be a finite simplicial graph. Then for every integer $n \geq 0$, the Bestvina Brady group BB_Γ is of type \mathcal{F}_n if and only if the space X_Γ is $(n-1)$ -connected, that is, $\pi_i(X_\Gamma) = 1$ for all $0 \leq i \leq n$.*

Dimca, Papdima and Suciu showed that the only Bestvina-Brady groups which are Kähler are the obvious ones in the following sense

Theorem 3.2 ([6], Corollary 1.3). *Let Γ be a finite simplicial graph. Then BB_Γ is Kähler if and only if BB_Γ is free abelian of even rank.*

Free abelian groups are of type \mathcal{F}_∞ , since their classifying space is a torus. Hence, the Bestvina Brady groups can not provide any examples of Kähler groups with arbitrary finiteness properties. However, Dimca, Papadima and Suciu [7] observed that one can imitate the Bestvina Brady construction to obtain Kähler groups with arbitrary finiteness properties.

From a group theoretic point of view, their groups arise as kernels of maps from a direct product of surface groups to \mathbb{Z}^2 ; We will describe the geometric construction behind these maps later in this paper. Let $\Lambda_{g_1}, \dots, \Lambda_{g_r}$, $g_1, \dots, g_r \geq 2$ be surface groups with presentation

$$\Lambda_{g_i} = \langle a_1^i, \dots, a_{g_i}^i, b_1^i, \dots, b_{g_i}^i \mid [a_1^i, b_1^i] \cdots [a_{g_i}^i, b_{g_i}^i] \rangle$$

Although they did not describe them as such, the Dimca, Papadima and Suciu groups are kernels of the surjective maps

$$\begin{aligned} \phi_{g_1, \dots, g_r} : \Lambda_{g_1} \times \cdots \times \Lambda_{g_r} &\rightarrow \mathbb{Z}^2 = \langle a, b \mid [a, b] \rangle \\ a_1^i, a_2^i &\mapsto a \\ b_1^i, b_2^i &\mapsto b \\ a_3^i, \dots, a_{g_i}^i &\mapsto 0 \\ b_3^i, \dots, b_{g_i}^i &\mapsto 0, \end{aligned} \tag{3.1}$$

as our explanation of their construction in Section 4 will show, where $g_1, \dots, g_r \geq 2$ and $r \geq 3$.

In particular, we will see that the maps constructed in Section 4 satisfy all the conditions of Theorem 1.1. This will provide us with a new proof that the groups ϕ_{g_1, \dots, g_r} are Kähler groups of finiteness type \mathcal{F}_{r-1} , but not of type \mathcal{F}_r . Thus:

Theorem 3.3. *If $g_1, \dots, g_r \geq 2$, then the kernel of ϕ_{g_1, \dots, g_r} is a Kähler group of finiteness type \mathcal{F}_{r-1} , but not of type \mathcal{F}_r .*

The case when $g_1 = \dots = g_r = 2$ seems to be a completely analogous surface group version of the Bestvina Brady group corresponding to the direct $F_2 \times \cdots \times F_2$ of r copies of the free group on two generators.

More generally consider the direct product $\Lambda_{g_1} \times \cdots \times \Lambda_{g_r}$ for $g_1, \dots, g_r \geq 2$. Then, in analogy to the map $\phi_{2, \dots, 2}$, define the homomorphism

$$\begin{aligned} \psi_{g_1, \dots, g_r} : \Lambda_{g_1} \times \cdots \times \Lambda_{g_r} &\rightarrow \mathbb{Z}^2 = \langle a, b \mid [a, b] \rangle \\ a_j^i &\mapsto a \\ b_j^i &\mapsto b \end{aligned} \tag{3.2}$$

To prove that all groups arising as kernel of one of the ψ_{g_1, \dots, g_r} are not of type \mathcal{F}_r , we use a result by Bridson, Miller, Howie and Short [3, Theorem B]

Theorem 3.4. *Let $\Lambda_{g_1}, \dots, \Lambda_{g_n}$ be surface groups and let $G \leq \Lambda_{g_1} \times \cdots \times \Lambda_{g_n}$ be a subgroup of their direct product. Assume that each intersection $L_i = G \cap \Lambda_{g_i}$ is non-trivial and arrange the factors in such a way that L_1, \dots, L_r are not finitely generated and L_{r+1}, \dots, L_n are finitely generated.*

If precisely $r \geq 1$ of the L_i are not finitely generated, then G is not of type FP_r .

For finitely presented groups, being of type FP_r is equivalent to being of type \mathcal{F}_r [5, p. 197]. Since all the groups we consider will be finitely presented, we do not dwell on the meaning of the homological finiteness condition FP_r . As a consequence of Theorem 3.4 we can now prove

Theorem 3.5. *Let $\Lambda_{g_1}, \dots, \Lambda_{g_r}$ be surface groups, $r \geq 3$. Let $k \geq 1$ and let $f_i : \Lambda_{g_i} \rightarrow \mathbb{Z}^k$ be non-trivial homomorphisms. Then the kernel of the map*

$$f = f_1 + \dots + f_r : \Lambda_{g_1} \times \dots \times \Lambda_{g_r} \rightarrow \mathbb{Z}^k$$

is not of type \mathcal{F}_r

Proof. Let $G = \ker(f) < \Lambda_{g_1} \times \dots \times \Lambda_{g_r}$. Let $L_i = G \cap \Lambda_{g_i} := G \cap 1 \times \dots \times \Lambda_{g_i} \times \dots \times 1$ and note that $L_i = \ker(f_i)$. Then L_i is an infinite index normal subgroup of Λ_{g_i} . It is well-known that all infinite index normal subgroups of a surface group are infinitely generated free groups. Hence, none of the L_i are finitely generated and therefore G is not of type FP_r , by Theorem 3.4, and in particular not of type \mathcal{F}_r . \square

In particular this gives an alternative proof of the finiteness properties of the Kähler groups constructed by Dimca, Papadima and Suciu. In fact, we immediately obtain

Corollary 3.6. *For all $r \geq 3$ and $g_1, \dots, g_r \geq 2$, the groups $\ker(\phi_{g_1, \dots, g_r})$ and $\ker(\psi_{g_1, \dots, g_r})$ are not of type \mathcal{F}_r .*

Remark 3.7. Under the additional condition that the maps $f_i : \Lambda_i \rightarrow \mathbb{Z}^k$ are surjective all groups that arise in this way are group theoretic fibre products over \mathbb{Z}^k , one can construct explicit finite presentations for them using the same methods as in [9].

4. CONSTRUCTING LARGE CLASSES OF EXAMPLES

In this section we will show how to use our results to prove Theorem 1.1.

Let E be a closed Riemann surface of positive genus and let X be a closed connected complex analytic manifold of dimension $r > 1$. An *irrational pencil* $h : X \rightarrow E$ is a surjective holomorphic map such that the smooth generic fiber H is connected.

For M, N complex manifolds, we say that a surjective holomorphic $f : M \rightarrow N$ has *isolated singularities* if, for every $y \in N$, the set of singular points of f in $f^{-1}(y)$ is discrete. In the case of an irrational pencil $h : X \rightarrow E$ this is equivalent to the set of singular points of f being finite.

The following result is due to Dimca, Papadima and Suciu

Theorem 4.1 ([7], Theorem C). *Let $h : X \rightarrow E$ be an irrational pencil. Suppose that h has only isolated singularities. Then the following hold:*

- (1) *The inclusion $H \hookrightarrow X$ induces isomorphisms $\pi_i(H) \cong \pi_i(X)$ for $2 \leq i \leq r-1$*
- (2) *h induces a short exact sequence $1 \rightarrow \pi_1 H \rightarrow \pi_1 X \rightarrow \pi_1 E \rightarrow 1$*

A *Stein manifold* is a complex manifold that embeds biholomorphically as a closed submanifold in some affine complex space \mathbb{C}^r . Theorem 4.1 allows us to prove

Theorem 4.2. *Let $r \geq 3$ and let $g_1, \dots, g_r \geq 2$. Let E be an elliptic curve and, for $i = 1, \dots, r$, let S_{g_i} be a closed connected surface of genus g_i together with a branched covering $f_{g_i} : S_{g_i} \rightarrow E$. Assume further that the map*

$$f = \sum_{i=1}^r f_{g_i} : S_{g_1} \times \dots \times S_{g_r} \rightarrow E$$

induces a surjective map on fundamental groups.

Then the generic fibre H of the map f is a connected $(r-1)$ -dimensional smooth projective variety such that

- (1) The homotopy groups $\pi_i H$ are trivial for $2 \leq i \leq r-2$ and $\pi_{r-1} H$ is nontrivial.
- (2) The universal cover \tilde{H} of H is a Stein manifold
- (3) The fundamental group $\pi_1 H$ is a projective (and thus Kähler) group of finiteness type \mathcal{F}_{r-1} , but not of finiteness type \mathcal{F}_r .
- (4) The fundamental group $\pi_1 H$ is not commensurable (up to finite kernels) to any group having a classifying space of finite type
- (5) The map f induces a short exact sequence $1 \rightarrow \pi_1 H \rightarrow \pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_r} \rightarrow \pi_1 E \rightarrow 1$ on fundamental groups.

Proof. It is well-known that there is a unique complex structure on S_{g_i} with respect to which the map f_{g_i} is holomorphic, since f_{g_i} is a finite-sheeted branched covering map. In particular, f_{g_i} has critical points the finite preimage $C_i = f_{g_i}^{-1}(D_i)$, where $D_i \subset E$ is the finite set of branching points of f_{g_i} .

This equips $X = S_{g_1} \times \cdots \times S_{g_r}$ with a projective structure with respect to which $f : X = S_{g_1} \times \cdots \times S_{g_r} \rightarrow E$ is a holomorphic submersion. The set of singular points of f is then the set of points $(x_1, \dots, x_r) \in S_{g_1} \times \cdots \times S_{g_r}$ such that $0 = df(x_1, \dots, x_r) = (df_{g_1}(x_1), \dots, df_{g_r}(x_r))$. It follows that the set of singular points of f is the finite set $C_1 \times \cdots \times C_r$. In particular, f has isolated singularities.

By assumption all of the f_{g_i} are branched covering maps and f is surjective on fundamental groups. Thus, by Theorem 2.1, the map f has connected fibers.

It follows that f is an irrational pencil with isolated singularities. Hence, by Theorem 4.1, we obtain that f induces a short exact sequence

$$1 \rightarrow \pi_1 H \rightarrow \pi_1 X \xrightarrow{f_{\#}} \pi_1 E \rightarrow 1$$

on fundamental groups proving assertion (5). Furthermore, we obtain that $\pi_i H \cong \pi_i X \cong 0$, for $2 \leq i \leq r-1$, where the last equality follows since X is a $K(\pi_1 X, 1)$. This implies the first part of assertion (1).

The group $\pi_1 H$ is projective, since the generic smooth fibre H of f is a complex submanifold of the compact projective manifold X .

Because $\pi_1 H = 0$ for $2 \leq i \leq r-1$, we obtain a $K(\pi_1 H, 1)$ from H by attaching cells of dimension $\geq r$. Since H is a compact complex manifold, it follows that H has a finite cell structure. Thus, the group $\ker(f_{\#}) = \pi_i H$ is of finiteness type \mathcal{F}_{r-1} and, by Theorem 3.5, it is not of type \mathcal{F}_r . This implies assertion (3) and the second part of assertion (1), since if $\pi_{r-1} H$ was trivial we could construct a $K(G, 1)$ with finite r -skeleton.

Assertion (4) is an immediate consequence of the well-known [7, Proposition 2.7].

Assertion (1) follows similarly as in the proof of [7, Theorem A]. Namely, the universal covering \tilde{X} , $q : \tilde{X} \rightarrow X$ of $X = S_{g_1} \times \cdots \times S_{g_r}$ is a contractible Stein manifold and the pair (X, H) is $(r-1)$ -connected by Theorem 4.1. Hence, the preimage $q^{-1}(H)$ of H in \tilde{X} is a closed complex submanifold of the Stein manifold \tilde{X} which is biholomorphic to the universal covering \tilde{H} of H . Thus, \tilde{H} is Stein. \square

Theorem 1.1 is now a direct consequence of Theorem 4.2. Note that Theorem 4.2 is a generalisation of [7, Theorem A].

5. CONSTRUCTING A BESTVINA BRADY TYPE CLASS OF EXAMPLES

We will now proceed to explain how one can realise the maps ϕ_{g_1, \dots, g_r} in (3.1) and ψ_{g_1, \dots, g_r} in (3.2) geometrically. More precisely, they will arise as the induced maps on fundamental groups of maps satisfying the conditions of Theorem 1.1.

The map realising ϕ_{g_1, \dots, g_r} is the map used for the original construction of Kähler groups with arbitrary finiteness properties by Dimca, Papadima and Suciu. Since our construction imitates the construction by Dimca, Papadima and Suciu, we want to give a brief description of their construction first.

Let E be an elliptic curve, that is, a complex torus of dimension one, let $g \geq 2$, and let $B = \{b_1, \dots, b_{2g-2}\} \subset E$ be a finite subset of even size. Choose a set of generators $\alpha, \beta, \gamma_1, \dots, \gamma_{2g-2}$ of the first homology group $H_1(E \setminus B, \mathbb{Z})$ such that γ_i is the boundary of a small disc centred at b_i , $i = 1, \dots, 2g-2$ and α, β is a basis of $\pi_1 E$.

Then the map $H_1(E \setminus B) \rightarrow \mathbb{Z}/2\mathbb{Z}$ defined by $\gamma_i \mapsto 1$ and $\alpha, \beta \mapsto 0$ induces a 2-fold normal covering of $\pi_1(E \setminus B)$ which extends continuously to a 2-fold branched covering $f_g : S_g \rightarrow E$ from a topological surface S_g of genus $g \geq 2$ onto E . It is well-known there is a unique complex structure on S_g such that the map f_g is holomorphic.

By looking at a suitable concrete realization of the map f_g (see [9] for more details), it is not hard to see that it induces the map

$$\begin{array}{lll} f_{g\#} : \Lambda_g = \pi_1 S_g & \rightarrow & \pi_1 E = \mathbb{Z}^2 \\ a_1, a_2 & \mapsto & a \\ b_1, b_2 & \mapsto & b \\ a_i, b_i & \mapsto & 0 \quad \forall i = 3, \dots, g \end{array}$$

with respect to presentations $\Lambda_g = \langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle$ and $\pi_1 E = \langle a, b \mid [a, b] \rangle$ of the fundamental groups.

Then for $g_1, \dots, g_r \geq 2$ and $r \geq 3$, use addition in the elliptic curve to define the map $f = \sum_{i=1}^r f_{g_i} : S_{g_1} \times \cdots \times S_{g_r} \rightarrow E$.

The maps f_{g_i} are branched coverings inducing surjective maps on fundamental groups. They restrict to unramified normal coverings of $E \setminus B_i$ which correspond to homomorphisms mapping a set of generators $\alpha, \beta : [0, 1] \rightarrow E \setminus B_i$ of $\pi_1 E$ to zero. It follows that f_{g_i} is purely branched for $i = 1, \dots, r$. In particular they are surjective on fundamental groups.

Hence, by Theorem 1.1, the kernel of the induced map $f_{\#} : \Lambda_{g_1} \times \cdots \times \Lambda_{g_r} \rightarrow \mathbb{Z}^2$ is projective (and thus Kähler) of finiteness type \mathcal{F}_{r-1} , but not \mathcal{F}_r . Observe that the map $f_{\#}$ is indeed identical with the map ϕ_{g_1, \dots, g_r} defined in (3.1).

We will imitate this construction in order to produce holomorphic maps $g_{h_1}, \dots, g_{h_r} : S_g \rightarrow E$ for all $g_1, \dots, g_r \geq 2$ and $r \geq 3$ such that $g = \sum_{i=1}^r g_i$ realizes the map ψ_{h_1, \dots, h_r} defined in (3.2). We will present two different constructions, each of which has different advantages.

Construction 1: This is the more natural construction. It has the advantage that the maps g_{h_i} are normal branched coverings, but it comes at the cost that the singularities of g are not quadratic and therefore g is not a Morse function.

As above, let E be an elliptic curve and $B = \{d_1, d_2\} \subset E$ two arbitrary points. Let $\alpha, \beta, \gamma_1, \gamma_2$ be the same set of generators for the homology group $H_1(E \setminus B, \mathbb{Z})$ as above, that is, α, β are generators of $\pi_1 E$ intersecting positively with respect to the orientation induced by the complex structure on E and γ_1, γ_2 are the positively oriented boundary loops of small discs around b_1 , respectively b_2 . For $h \geq 2$, the surjective homomorphism $H_1(E \setminus B, \mathbb{Z}) \rightarrow \mathbb{Z}/h\mathbb{Z}$ defined by $\alpha, \beta \mapsto 0, \gamma_1 \mapsto 1, \gamma_2 \mapsto -1$, defines a h -fold normal branched covering $g_h : S_h \rightarrow E$ from a topological surface of genus h with branching locus B . We may assume that, after connecting the generators of $H_1(E \setminus B, \mathbb{Z})$ to a base point, the fundamental group of $\pi_1(E \setminus B)$ is equal to

$$\pi_1 E \setminus B = \langle \alpha, \beta, \gamma_1, \gamma_2 \mid [\alpha, \beta] \gamma_1 \gamma_2 \rangle$$

Again it is well-known that there is a unique complex structure on S_h such that the map g_h is holomorphic. Denote by $C = \{c_1 = g_h^{-1}(d_1), c_2 = g_h^{-1}(d_2)\} \subset S_h$ the set of critical points of g_h .

In analogy to Dimca, Papadima and Suciu's groups we go on to define the map g using the additive structure on E by

$$g = \sum_{i=1}^r g_{h_i} : S_{h_1} \times \cdots \times S_{h_r} \rightarrow E$$

for all $r \geq 3$ and $h_1, \dots, h_r \geq 2$.

The maps g_{h_i} are branched coverings induced by the surjective composition of homomorphisms $\pi_1(E \setminus B) \rightarrow H_1(E \setminus B, \mathbb{Z}) \rightarrow \mathbb{Z}/h_i\mathbb{Z}$ defined by $\alpha, \beta \mapsto 0, \gamma_1 \mapsto 1, \gamma_2 \mapsto -1$. In particular, all of the g_{h_i} are purely branched, since α and β are elements of the kernel of this homomorphism, which is a normal subgroup of $\pi_1(E \setminus B)$. It follows that all of the g_{h_i} are surjective on fundamental groups.

Hence, Theorem 1.1 can be applied to g . This implies that the fundamental group $\pi_1 H$ of the generic fibre H of g is a projective (and thus Kähler) group of finiteness type \mathcal{F}_{r-1} , but not of finiteness type \mathcal{F}_r .

Since all of the g_{h_i} are purely branched, any lift of the generators $\alpha, \beta : [0, 1] \rightarrow E \setminus B_i$ of $\pi_1 E$ to S_{h_i} is a loop. Choose a fundamental domain $F \subset S_{h_i}$ for the $\mathbb{Z}/h_i\mathbb{Z}$ -action such that the images of α and β are contained in the image $g_{h_i}(U)$ of an open subset $U \subset F$ on which g_{h_i} restricts to a homeomorphism.

Denote the h_i lifts of α by $a_1^{(i)}, \dots, a_{h_i}^{(i)}$ and the h_i lifts of β by $b_1^{(i)}, \dots, b_{h_i}^{(i)}$, where we choose lifts so that $a_j^{(1)}$ and $b_j^{(1)}$ are in the interior of the fundamental domain $\bar{j} \cdot F$ for $\bar{j} \in \mathbb{Z}/h_i\mathbb{Z}$. In particular, the loops $a_1^{(i)}, b_1^{(i)}, \dots, a_{h_i}^{(i)}, b_{h_i}^{(i)}$ form a standard symplectic basis for the (symplectic) intersection form on $H_1(S_{h_i}, \mathbb{Z})$.

It is then well-known that we can find generators $\alpha_1^{(i)}, \beta_1^{(i)}, \dots, \alpha_{h_i}^{(i)}, \beta_{h_i}^{(i)}$ of $\pi_1 S_{h_i}$ such that the abelianization is given by $\alpha_j^{(i)} \mapsto a_j^{(i)}, \beta_j^{(i)} \mapsto b_j^{(i)}$ and

$$\pi_1 S_{h_i} = \left\langle \alpha_1^{(i)}, \beta_1^{(i)}, \dots, \alpha_{h_i}^{(i)}, \beta_{h_i}^{(i)} \mid [\alpha_1^{(i)}, \beta_1^{(i)}] \cdots [\alpha_{h_i}^{(i)}, \beta_{h_i}^{(i)}] \right\rangle.$$

This is for instance an easy consequence of Theorem 6.5.

With respect to this presentation, the map on fundamental groups induced by g_{h_i} is given by

$$\begin{array}{ccc} g_{h_i\#} : & \pi_1 S_{h_i} & \rightarrow \pi_1 E \\ & \alpha_j^{(i)} & \mapsto \alpha \\ & \beta_j^{(i)} & \mapsto \beta \end{array}$$

For an illustration of the map g_h and the generators α_j, β_j , see Figure 1.

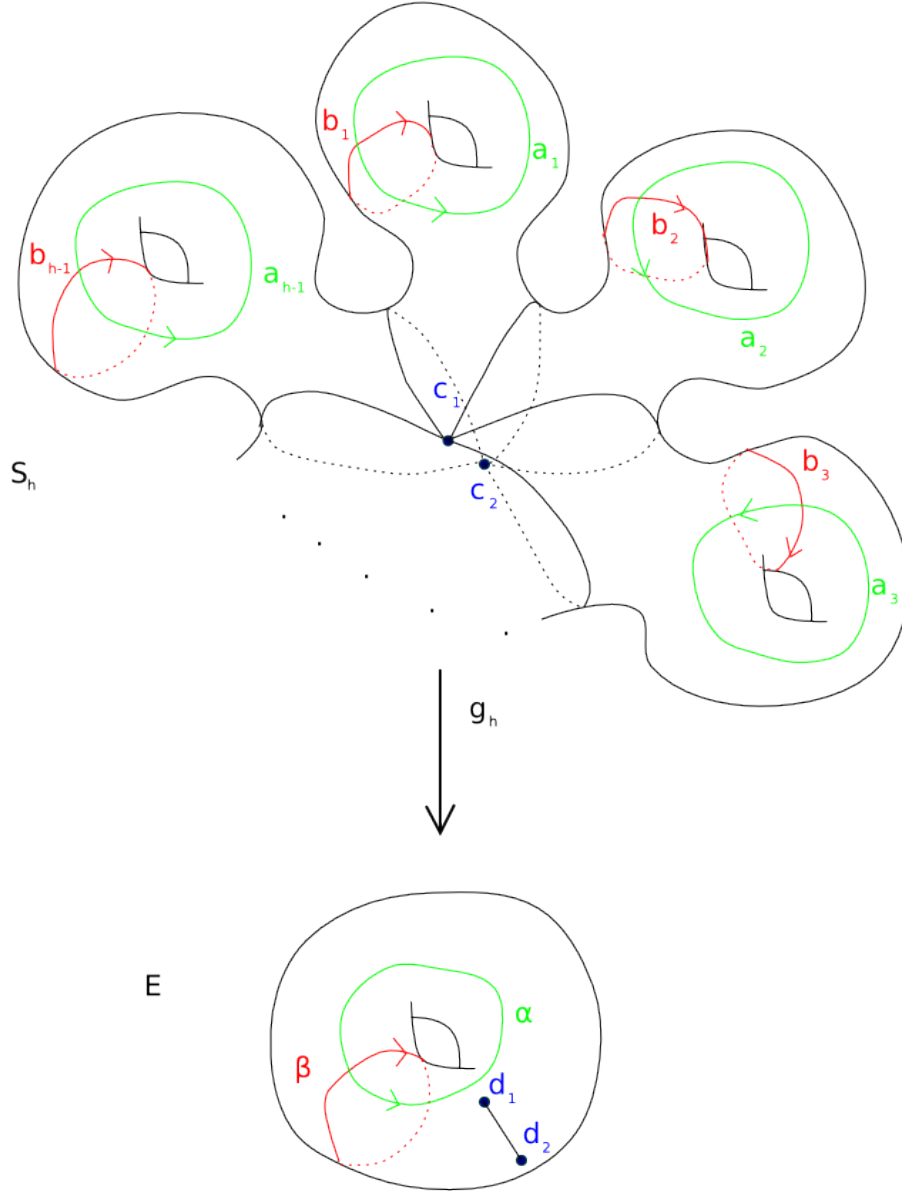


FIGURE 1. The h -fold branched normal covering g_h of E in Construction 1

As a direct consequence, we obtain that g induces the map

$$\begin{aligned}
 g_{\#} : \pi_1 S_{h_1} \times \cdots \times \pi_1 S_{h_r} &\rightarrow \pi_1 E \\
 \alpha_j^{(i)} &\mapsto \alpha \\
 \beta_j^{(i)} &\mapsto \beta
 \end{aligned}$$

on fundamental groups. Thus, the induced map $g_{\#}$ on fundamental groups is indeed the map ψ_{h_1, \dots, h_r} in (3.2).

Construction 2: We will now give an alternative construction which realises ψ_{h_1, \dots, h_r} as the fundamental group of the generic fibre of a fibration over an elliptic curve with Morse type

singularities only. This is at the expense of the maps g_{h_i} being regular branched coverings rather than normal branched coverings.

Let E be an elliptic curve, let $h \geq 2$, let $d_{1,1}, d_{1,2}, d_{2,1}, \dots, d_{h-1,1}, d_{h-1,2}$ be $2(h-1)$ points in E and let $s_1, \dots, s_{h-1} : [0, 1] \rightarrow E$ be simple, pairwise non-intersecting paths with starting point $s_i(0) = d_{i,1}$ and endpoint $s_i(1) = d_{i,2}$ for $i = 1, \dots, h-1$. Take g copies E_0, E_1, \dots, E_{h-1} of E , cut E_0 open along all of the paths s_i and cut E_i open along the path s_i for $1 \leq i \leq h-1$. This produces surfaces F_0, \dots, F_{h-1} with boundary.

Glue the boundary of F_i to the boundary component of F_0 corresponding to the cut produced by the path s_i where we identify opposite edges with respect to the identity homeomorphism $E_0 \rightarrow E_i$ for $i = 1, \dots, h-1$. This yields a closed genus h surface R_h together with a $g-1$ -fold branched covering map $g'_h : R_h \rightarrow E$ with branching points $b_{1,1}, b_{1,2}, b_{2,1}, \dots, b_{h-1,1}, b_{h-1,2}$ of order two. Endow R_h with the unique complex structure that makes the map g'_h holomorphic.

It is clear that the map g'_h is purely branched and surjective on fundamental groups. In analogy to Construction 1 we find a standard generating sets α, β of $\pi_1 E$ and $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ of $\pi_1 S_g$ with respect to which the induced map on fundamental groups is given by $\alpha_i \mapsto \alpha$ and $\beta_i \mapsto \beta$. See figure 2 for an illustration of the map g'_h .

For $h_1, \dots, h_r \geq 2$, $r \geq 3$, the holomorphic map $g' = \sum_{i=1}^r g'_{h_i} : R_{h_1} \times \dots \times R_{h_r}$ induces the map ψ_{h_1, \dots, h_r} on fundamental groups. The map g' has isolated singularities and connected fibres and in fact we can see, by considering local coordinates around the singular points, that all singularities of g' are of Morse type.

6. REDUCING THE ISOMORPHISM TYPE OF OUR EXAMPLES TO LINEAR ALGEBRA

We will now show that our groups are not isomorphic to Dimca, Papadima and Suciu's groups and thus provide genuinely new examples rather than being their examples in disguised form.

As before, let Λ_g be the fundamental group of a closed orientable surface of genus g . For $r, k \geq 1$, consider surjective homomorphisms $f_{g_i} : \Lambda_{g_i} \rightarrow \mathbb{Z}^k$ and $g_{h_i} : \Lambda_{h_i} \rightarrow \mathbb{Z}^k$, where $g_i, h_i \geq 2$ and $1 \leq i \leq r$. Recall that $\ker(f_{g_i})$ and $\ker(g_{h_i})$ are infinitely generated free groups, since they are infinite rank normal subgroups of surface groups.

Define maps $\phi_{g_1, \dots, g_r} = f_{g_1} + \dots + f_{g_r} : \Lambda_{g_1} \times \dots \times \Lambda_{g_r} \rightarrow \mathbb{Z}^k$ and $\psi_{h_1, \dots, h_r} = g_{h_1} + \dots + g_{h_r} : \Lambda_{h_1} \times \dots \times \Lambda_{h_r} \rightarrow \mathbb{Z}^k$ and let

$$L_i := \Lambda_{g_i} \cap \ker(\phi_{g_1, \dots, g_r}) = \ker(f_{g_i}),$$

$$K_i := \Lambda_{h_i} \cap \ker(\psi_{h_1, \dots, h_r}) = \ker(g_{h_i}).$$

Then the following Lemma is a special case of [4, Theorem C (3),(4)]

Lemma 6.1. *Every isomorphism*

$$\theta : \ker(\phi_{g_1, \dots, g_r}) \rightarrow \ker(\psi_{h_1, \dots, h_r})$$

satisfies $\theta(L_i) = K_i$ up to reordering of the factors. In particular, we obtain that θ restricts to an isomorphism $L_1 \times \dots \times L_r \cong K_1 \times \dots \times K_r$.

Furthermore, with the same reordering of factors, we have $\theta((\Lambda_{g_{i_1}} \times \dots \times \Lambda_{g_{i_k}}) \cap \ker(\phi_{g_1, \dots, g_r})) = (\Lambda_{h_{i_1}} \times \dots \times \Lambda_{h_{i_k}}) \cap \ker(\psi_{h_1, \dots, h_r})$ for $1 \leq k \leq r$ and $1 \leq i_1 < \dots < i_k \leq r$.

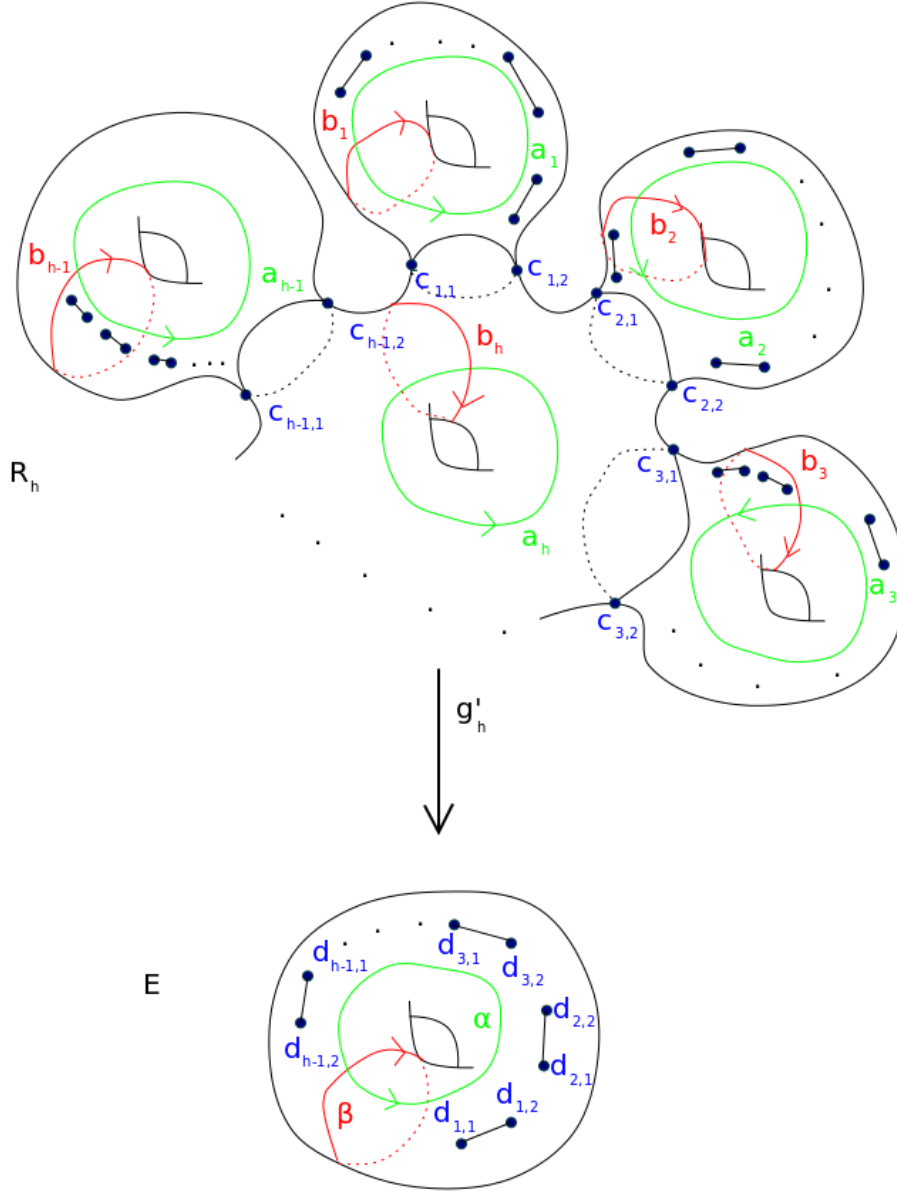


FIGURE 2. The h -fold branched covering g'_h of E with Morse type singularities in Construction 2

Remark 6.2. It also follows from [4, Theorem C (3),(4)] that if we have an isomorphism between direct products of r surface groups $\Lambda_{g_1} \times \cdots \times \Lambda_{g_r}$ and $\Lambda_{h_1} \times \cdots \times \Lambda_{h_r}$, then after reordering of the factors it is induced by isomorphisms $\Lambda_{g_i} \cong \Lambda_{h_i}$ and, in particular, $g_i = h_i$ for $i = 1, \dots, r$.

We obtain the following consequence

Theorem 6.3. *There is an isomorphism of the short exact sequences*

$$1 \rightarrow \ker(\phi_{g_1, \dots, g_r}) \rightarrow \Lambda_{g_1} \times \cdots \times \Lambda_{g_r} \xrightarrow{\phi_{g_1, \dots, g_r}} \mathbb{Z}^k \rightarrow 1$$

and

$$1 \rightarrow \ker(\psi_{h_1, \dots, h_r}) \rightarrow \Lambda_{h_1} \times \cdots \times \Lambda_{h_r} \xrightarrow{\psi_{h_1, \dots, h_r}} \mathbb{Z}^k \rightarrow 1$$

if and only if (up to reordering factors) $g_1 = h_1, \dots, g_r = h_r$ and there are isomorphisms of the short exact sequences

$$1 \rightarrow \ker(f_{g_i}) \rightarrow \Lambda_{g_i} \xrightarrow{f_{g_i}} \mathbb{Z}^k \rightarrow 1$$

and

$$1 \rightarrow \ker(g_{h_i}) \rightarrow \Lambda_{g_i} \xrightarrow{g_{h_i}} \mathbb{Z}^k \rightarrow 1$$

for all $i = 1, \dots, r$ such that the isomorphism $A: \mathbb{Z}^k \rightarrow \mathbb{Z}^k$ is independent of i .

Proof. The if direction follows immediately by taking the cartesian product of the automorphisms $\theta_i: \Lambda_{g_i} \rightarrow \Lambda_{g_i}$ to be the automorphism of $\Lambda_{g_1} \times \dots \times \Lambda_{g_r}$. and the identity map to be the automorphism of \mathbb{Z}^k .

For the only if direction we use that, by Remark 6.2, the isomorphism $\Lambda_{g_1} \times \dots \times \Lambda_{g_r} \rightarrow \Lambda_{h_1} \times \dots \times \Lambda_{h_r}$ is realised by a direct product of isomorphisms $\theta_i: \Lambda_{g_i} \rightarrow \Lambda_{h_i}$, after possibly reordering factors. Restricting to factors then implies that the automorphism θ_i of Λ_{g_i} and the identity on \mathbb{Z}^k induce an isomorphism of short exact sequences for $i = 1, \dots, r$. \square

Theorem 6.4. *The groups $H_1 = \ker(\phi_{g_1, \dots, g_r})$ and $H_2 = \ker(\psi_{h_1, \dots, h_s})$ are isomorphic if and only if there is an isomorphism of the short exact sequences*

$$1 \rightarrow \ker(\phi_{g_1, \dots, g_r}) \rightarrow \Lambda_{g_1} \times \dots \times \Lambda_{g_r} \xrightarrow{\phi_{g_1, \dots, g_r}} \mathbb{Z}^k \rightarrow 1$$

and

$$1 \rightarrow \ker(\psi_{h_1, \dots, h_s}) \rightarrow \Lambda_{h_1} \times \dots \times \Lambda_{h_s} \xrightarrow{\psi_{h_1, \dots, h_s}} \mathbb{Z}^k \rightarrow 1.$$

Proof. Let $\theta: H_1 \rightarrow H_2$ be an abstract isomorphism of groups and let $L_i \leq H_1$, $K_i \leq H_2$ be as above. By Lemma 6.1, we may assume that after reordering factors $\theta(L_i) \leq K_i$ and that $\theta(M_1) = M_2$ for $M_1 = H_1 \cap (1 \times \Lambda_{g_2} \times \dots \times \Lambda_{g_r})$, $M_2 = H_2 \cap (1 \times \Lambda_{h_2} \times \dots \times \Lambda_{h_s})$.

Since f_{g_i} , g_{h_j} are surjective for all $1 \leq i \leq r$, $1 \leq j \leq s$, we obtain that

$$\begin{aligned} H_1/M_1 &\cong \Lambda_{g_1}, & H_1/L_1 &\cong \Lambda_{g_2} \times \dots \times \Lambda_{g_r} \\ H_2/M_2 &\cong \Lambda_{h_1}, & H_2/K_1 &\cong \Lambda_{h_2} \times \dots \times \Lambda_{h_s}, \end{aligned}$$

where the isomorphisms are induced by the projection maps.

In particular, the map θ induces an isomorphism of short exact sequences

$$\begin{array}{ccccccc} 1 & \longrightarrow & K_1 & \longrightarrow & H_1/M_1 \times H_1/L_1 & \cong \Lambda_{g_1} \times \dots \times \Lambda_{g_r} & \longrightarrow \mathbb{Z}^k \longrightarrow 1 \\ \downarrow & & \theta \downarrow \cong & & \theta \downarrow \cong & & \bar{\theta} \downarrow \cong \\ 1 & \longrightarrow & K_2 & \longrightarrow & H_2/M_2 \times H_2/K_1 & \cong \Lambda_{h_1} \times \dots \times \Lambda_{h_s} & \longrightarrow \mathbb{Z}^k \longrightarrow 1 \end{array}$$

proving the only if direction. The if direction is trivial. \square

Note that Theorem 6.4 is well-known (see for instance [4, Section 7.5]).

It follows that in order to understand abstract isomorphisms of the kernels of maps of the form ϕ_{g_1, \dots, g_r} and ψ_{h_1, \dots, h_r} , it suffices to understand isomorphisms of short exact sequences of the form

$$1 \rightarrow N \rightarrow \Lambda_g \xrightarrow{f_g} \mathbb{Z}^k \rightarrow 1.$$

The latter reduces to Linear Algebra. We will explain this fact briefly. A detailed exposition of the subject can be found in [8] (see in particular Chapter 6). In the following let S_g be a closed surface of genus g and let $\Lambda_g = \pi_1(S_g)$ be its fundamental group.

Let $MCG^\pm(S_g)$ be the (*extended*) *mapping class group* of S_g , that is, the group of homeomorphisms of S_g up to homotopy equivalences, where by the extended mapping class group we mean that we allow orientation reversing homeomorphisms. Let $Inn(\Lambda_g)$ be the group of *inner automorphisms* of Λ_g , that is, automorphisms of the form $g \mapsto h^{-1}gh$ for a fixed $h \in \Lambda_g$ and let $Out(\Lambda_g) = Aut(\Lambda_g)/Inn(\Lambda_g)$ be the group of *outer automorphisms* of Λ_g .

The map $\Lambda_g \rightarrow \mathbb{Z}^k$ splits through the abelianization $H_1(\Lambda_g, \mathbb{Z})$ of Λ_g . Every automorphism τ of Λ_g induces an automorphism $\tau_* \in GL(2g, \mathbb{Z})$ of the abelianization $H_1(\Lambda_g, \mathbb{Z})$. Since inner automorphisms act trivially on $H_1(\Lambda_g, \mathbb{Z})$, the induced homomorphism $Aut(\Lambda_g) \rightarrow GL(2g, \mathbb{Z})$ splits through $Out(\Lambda_g)$.

By the Dehn-Nielsen-Baer Theorem (cf. [8, Theorem 8.1]) the natural map $MCG^\pm(S_g) \rightarrow Out(\Lambda_g)$ is an isomorphism. In particular, we can realise any element of $Out(\Lambda_g)$ by a (up to homotopy) unique homeomorphism of S_g . It follows that for $\tau \in Aut(\Lambda_g)$, the map $\tau_* \in GL(2g, \mathbb{Z})$ is realised by some homeomorphism $\alpha \in MCG^\pm(S_g)$.

There is a natural symplectic form on $H_1(\Lambda_g, \mathbb{Z})$ induced by taking intersection numbers of representatives in S_g . Orientation preserving homeomorphism $\alpha \in MCG^+(S_g)$ of S_g preserve intersection numbers. Hence, the induced automorphism $\alpha_* \in H_1(\Lambda_g, \mathbb{Z})$ preserves the symplectic form which is equivalent to $A := \alpha_* \in Sp(2g, \mathbb{Z})$, where $Sp(2g, \mathbb{Z})$ is the group of *symplectic matrices* of dimension $2g$ with integer coefficients. It is defined by

$$Sp(2g, \mathbb{Z}) = \{A \in M_{2g}(\mathbb{Z}) \mid A^t J_{2g} A = J_{2g}\},$$

for J_{2g} the matrix representing the standard symplectic form given by the block diagonal matrix

$$J_{2g} = \begin{pmatrix} J & 0 & \cdots & 0 \\ 0 & J & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & J \end{pmatrix}$$

with $J_2 = J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ the standard symplectic form on \mathbb{R}^2 .

For an orientation reversing homeomorphism we have that $A^t J A = -J$. We define the generalized symplectic group of dimension $2g$ with integer coefficients by

$$Sp^\pm(2g, \mathbb{Z}) = \{A \in M_{2g}(\mathbb{Z}) \mid A^t J A = J \text{ or } A^t J A = -J\}.$$

As a result, there is a natural homomorphism $\Psi : MCG^\pm(S_g) \rightarrow Sp^\pm(2g, \mathbb{Z})$.

Theorem 6.5 ([8], Theorem 6.4). *The symplectic representation $\Psi : MCG^\pm(S_g) \rightarrow Sp^\pm(2g, \mathbb{Z})$ is surjective for $g \geq 1$.*

Combining this with the isomorphism $MCG^\pm(S_g) \rightarrow Out(S_g)$, we obtain

Corollary 6.6. *For $g \geq 1$, the symplectic representation Ψ induces a surjective representation $\text{Out}(S_g) \rightarrow \text{Sp}^\pm(2g, \mathbb{Z})$.*

In particular, two short exact sequences

$$1 \rightarrow \ker(f) \rightarrow \Lambda_g \xrightarrow{f} \mathbb{Z}^k \rightarrow 1$$

and

$$1 \rightarrow \ker(h) \rightarrow \Lambda_g \xrightarrow{h} \mathbb{Z}^k \rightarrow 1$$

are isomorphic if and only if there exists $A \in \text{Sp}^\pm(2g, \mathbb{Z})$ and $A' \in \text{GL}(k, \mathbb{Z})$ such that the following diagram commutes

$$\begin{array}{ccc} H_1(\Lambda_g, \mathbb{Z}) & \xrightarrow{f_{ab}} & \mathbb{Z}^k \\ \downarrow A & & \downarrow A' \\ H_1(\Lambda_g, \mathbb{Z}) & \xrightarrow{g_{ab}} & \mathbb{Z}^k \end{array}$$

where f_{ab} and g_{ab} are the unique maps factoring f and g through their abelianization.

Proof. The first part is a direct consequence of Theorem 6.5 and the isomorphism $\text{MCG}^\pm(S_g) \cong \text{Out}(\Lambda_g)$.

The only if in the second part follows directly from the fact that for any outer automorphism of Λ_g , the induced map on homology is in the generalized symplectic group. The if follows, since we can lift any $A \in \text{Sp}(2g, \mathbb{Z})$ to an element $\alpha \in \text{Aut}(\Lambda_g)$ inducing A by the first part. \square

7. CLASSIFICATION FOR PURELY BRANCHED MAPS

It follows from Section 6, showing that our groups are not isomorphic to Dimca, Papadima and Suciu's groups amounts to comparing maps on abelianizations and therefore reduces to a question in Linear Algebra. In fact, it is possible to classify all maps on fundamental groups that arise from purely branched maps. As a Corollary of this classification we obtain that our groups are different from Dimca, Papadima and Suciu's groups.

Theorem 7.1. *Let S_g be a closed surface of genus g , let E be an elliptic curve and let $f_g : S_g \rightarrow E$ be a purely branched covering map. Then the following are equivalent:*

- (1) *There is a standard symplectic generating set $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ of $\pi_1 S_g$ and α, β of $\pi_1 E$ with respect to which the induced map on fundamental groups is*

$$\begin{array}{lll} f_{g\#} : & \pi_1 S_g & \rightarrow \pi_1 E \\ & \alpha_1, \dots, \alpha_k & \mapsto \alpha \\ & \beta_1, \dots, \beta_k & \mapsto \beta \\ & \alpha_{k+1}, \dots, \beta_g & \mapsto 0 \\ & \beta_{k+1}, \dots, \beta_g & \mapsto 0 \end{array}$$

- (2) *The map f_g is a k -fold purely branched covering map for $2 \leq k \leq g$*

Furthermore, for $2 \leq k \leq g$, there is a purely branched k -fold holomorphic covering map $f_g : S_g \rightarrow E$ with Morse type singularities.

As a consequence we can prove Theorem 1.2 giving a complete classification of all Kähler groups with arbitrary finiteness properties that arise from our construction in the case when all of the maps are purely branched

Proof of Theorem 1.2. This is a direct consequence of Theorem 7.1, Theorem 6.4 and Theorem 1.1. \square

It follows that our groups are indeed not isomorphic to Dimca, Papadima, and Suciu's groups apart from the obvious isomorphism for $r = s$ and $g_1 = \dots = g_r = h_1 = \dots = h_r = 2$:

Corollary 7.2. *The Kähler groups $\ker(\phi_{g_1, \dots, g_r})$ obtained from (3.1) and $\ker(\psi_{h_1, \dots, h_r})$ obtained from (3.2) are isomorphic if and only if $r = s$ and $g_1 = \dots = g_r = h_1 = \dots = h_r = 2$.*

Proof. This is an immediate consequence of Theorem 1.2 and the fact that we constructed our groups as fundamental groups of the fibre of a sum of h_i -fold purely branched holomorphic maps in Section 5, while, as we also saw in Section 5, Dimca, Papadima and Suciu's groups arise as fundamental groups of the fibre of a sum of 2-fold purely branched holomorphic maps. \square

The part of Theorem 7.1 that for $2 \leq k < l \leq g$ we obtain distinct maps on fundamental groups will follow from Section 6 and the following result in Linear Algebra

Proposition 7.3. *For $g \geq 2$ and $1 \leq k < l \leq g$ there are no linear maps $A \in Sp^\pm(2g, \mathbb{R})$ and $B \in Gl(2, \mathbb{Z}) = Sp^\pm(2, \mathbb{Z})$ such that*

$$\left(\underbrace{I \dots I}_{k \text{ times } g-k \text{ times}} \quad \underbrace{0 \dots 0}_{l \text{ times } g-l \text{ times}} \right) \cdot A = B \cdot \left(\underbrace{I \dots I}_{l \text{ times } g-l \text{ times}} \quad \underbrace{0 \dots 0}_{k \text{ times } g-k \text{ times}} \right), \quad (7.1)$$

where $I = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the 2-dimensional identity matrix.

Proof. The proof is by contradiction. Assume that there is $A \in Sp^\pm(2g, \mathbb{R})$ and $B \in Gl(2, \mathbb{Z}) = Sp^\pm(2, \mathbb{Z})$ satisfying Equation (7.1). Define $\gamma_1, \dots, \gamma_{2g} \in \mathbb{R}^{2k}$ and $\alpha_1, \dots, \alpha_{2g} \in \mathbb{R}^{2g-2k}$ by

$$A = \begin{pmatrix} \gamma_1 & \gamma_2 & \dots & \gamma_{2g} \\ \alpha_1 & \alpha_2 & \dots & \alpha_{2g} \end{pmatrix}.$$

Then $A \in Sp^\pm(2g, \mathbb{R})$ implies that

$$\begin{aligned} \pm J_{2g} &= A^t J A = \begin{pmatrix} \gamma_1^t & \alpha_1^t \\ \vdots & \vdots \\ \gamma_{2g}^t & \alpha_{2g}^t \end{pmatrix} \cdot \begin{pmatrix} J_{2k} & 0 \\ 0 & J_{2g-2k} \end{pmatrix} \cdot \begin{pmatrix} \gamma_1 & \gamma_2 & \dots & \gamma_{2g} \\ \alpha_1 & \alpha_2 & \dots & \alpha_{2g} \end{pmatrix} \\ &= \begin{pmatrix} \gamma_1^t \\ \vdots \\ \gamma_{2g}^t \end{pmatrix} \cdot J_{2k} \cdot (\gamma_1 \dots \gamma_{2g}) + \begin{pmatrix} \alpha_1^t \\ \vdots \\ \alpha_{2g}^t \end{pmatrix} \cdot J_{2g-2k} \cdot (\alpha_1 \dots \alpha_{2g}) \\ &= \underbrace{[\gamma_i^t \cdot J_{2k} \cdot \gamma_j]_{i,j=1, \dots, 2g}}_{=:E} + \underbrace{[\alpha_i^t \cdot J_{2g-2k} \cdot \alpha_j]_{i,j=1, \dots, 2g}}_{=:F} \end{aligned}$$

The map E is of rank $\leq 2k$, since it splits through \mathbb{R}^{2k} and the map F is of rank $\leq 2g - 2k$, since it splits through \mathbb{R}^{2g-2k} .

We will now prove that in fact E is of rank $\leq 2k - 2$. Equation (7.1) implies that for

$$(\gamma_1 \dots \gamma_{2g}) = \begin{pmatrix} A_{11} & \dots & A_{1g} \\ \vdots & \ddots & \vdots \\ A_{k1} & \dots & A_{kg} \end{pmatrix},$$

where $A_{ij} \in \mathbb{R}^{2 \times 2}$ for $1 \leq i \leq k$ and $1 \leq j \leq g$, we have

$$\underbrace{(I \cdots I)}_{k \text{ times}} \cdot \begin{pmatrix} A_{11} & \cdots & A_{1g} \\ \vdots & \ddots & \vdots \\ A_{k1} & \cdots & A_{kg} \end{pmatrix} = \underbrace{(B \cdots B)}_{1 \text{ times}} \underbrace{(0 \cdots 0)}_{g-1 \text{ times}}.$$

It follows that

$$\sum_{i=1}^k A_{ij} = \begin{cases} B & \text{if } j \leq l \\ 0 & \text{if } j > l \end{cases}$$

and hence, that

$$(\gamma_1 \cdots \gamma_{2g}) = \begin{pmatrix} A_{11} & \cdots & A_{1l} & A_{1(l+1)} & \cdots & A_{1g} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{(k-1)1} & \cdots & A_{(k-1)l} & A_{(k-1)(l+1)} & \cdots & A_{(k-1)g} \\ B - \sum_{i=1}^{k-1} A_{i1} & \cdots & B - \sum_{i=1}^{k-1} A_{il} & -\sum_{i=1}^{k-1} A_{i(l+1)} & \cdots & -\sum_{i=1}^{k-1} A_{ig} \end{pmatrix} =: (MN)$$

with $M \in \mathbb{R}^{2k \times 2l}$ and $N \in \mathbb{R}^{2k \times 2(g-l)}$.

Then, we have

$$E = \begin{pmatrix} M^t J_{2k} M & M^t J_{2k} N \\ N^t J_{2k} M & N^t J_{2k} N \end{pmatrix}.$$

Define linear maps

$$M' = \left[\det B \cdot \left(B^{-1} A_{ij} - \frac{1}{\sqrt{k}-1} \sum_{l=1}^{k-1} B^{-1} A_{lj} + \frac{I_2}{\sqrt{k}} \right) \right]_{i=1, \dots, k-1, j=1, \dots, l} \in \mathbb{R}^{2(k-1) \times 2l}$$

$$N' = \left[\det B \cdot \left(B^{-1} A_{im} - \frac{1}{\sqrt{k}-1} \sum_{r=1}^{k-1} B^{-1} A_{rm} \right) \right]_{i=1, \dots, k-1, m=l+1, \dots, g} \in \mathbb{R}^{2(k-1) \times 2(g-l)}.$$

Using that $B^t J_2 B = \det B \cdot J_2$, we obtain the following identities

$$M^t J_{2k} M = \frac{\det B}{k} \cdot \begin{pmatrix} J & \cdots & J \\ \vdots & \ddots & \vdots \\ J & \cdots & J \end{pmatrix} + M'^t J_{2k-2} M'$$

$$M^t J_{2k} N = M'^t J_{2k-2} N'$$

$$N^t J_{2k} M = N'^t J_{2k-2} M'$$

$$N^t J_{2k} N = N'^t J_{2k-2} N'$$

In particular, this implies that

$$E = [\gamma_i^t J_{2k} \gamma_j]_{i,j=1, \dots, 2g} = \begin{pmatrix} \frac{\det B}{k} J & \cdots & \frac{\det B}{k} J & 0 \\ \vdots & \ddots & \vdots & 0 \\ \frac{\det B}{k} J & \cdots & \frac{\det B}{k} J & 0 \\ 0 & & 0 & 0 \end{pmatrix} + \underbrace{\begin{pmatrix} M'^t \\ N'^t \end{pmatrix} \cdot J_{2k-2} \cdot (M' N')}_{=: S}.$$

The linear map S splits through \mathbb{R}^{2k-2} , implying that $\text{rank}(S) \leq 2k-2$. Furthermore, $\pm J_{2g} = E + F$ and $\det B = \pm 1$ imply that

$$F = \pm J_{2g} - E = \underbrace{\begin{pmatrix} (\pm 1 \pm \frac{1}{k})J & \pm \frac{1}{k}J & \cdots & \cdots & \pm \frac{1}{k}J \\ \pm \frac{1}{k}J & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \pm \frac{1}{k}J \\ \pm \frac{1}{k}J & \cdots & \cdots & \pm \frac{1}{k}J & (\pm 1 \pm \frac{1}{k})J \\ & & 0 & & \pm J_{2(g-l)} \end{pmatrix}}_{=:R} - S \quad (7.2)$$

Hence, F has $\text{rank} \geq 2g - (2k - 2) = 2(g - k) + 2$, since by Lemma 7.4 below R is invertible for $l \neq k$. This contradicts $\text{rank} F \leq 2(g - k)$, showing that there are no $A \in Sp^\pm(2g, \mathbb{R})$ and $B \in Gl(2, \mathbb{Z})$ satisfying (7.1). \square

Lemma 7.4. *For $l \in \mathbb{Z}$, $k \in \mathbb{R}$, the linear map R defined in (7.2) is invertible if and only if $l \neq \pm k$.*

Proof. Clearly, it suffices to prove that for $k \neq l$, the matrix

$$R_{2l} = \begin{pmatrix} (\pm 1 \pm \frac{1}{k})J & \pm \frac{1}{k}J & \cdots & \cdots & \pm \frac{1}{k}J \\ \pm \frac{1}{k}J & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \pm \frac{1}{k}J \\ \pm \frac{1}{k}J & \cdots & \cdots & \pm \frac{1}{k}J & (\pm 1 \pm \frac{1}{k})J \end{pmatrix} \in \mathbb{R}^{2l \times 2l}$$

is invertible. We do row and column operations in order to compute the rank of R_{2l} .

Subtracting the last (double) row from all other rows yields

$$\begin{pmatrix} \pm J & 0 & \cdots & 0 & \mp J \\ 0 & \pm J & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \pm J & \mp J \\ \pm \frac{1}{k}J & \cdots & \cdots & \pm \frac{1}{k}J & (\pm 1 \pm \frac{1}{k})J \end{pmatrix}$$

After subtracting multiples of the first $(l-1)$ (double) rows from the last row we obtain

$$\begin{pmatrix} \pm J & 0 & \cdots & 0 & \mp J \\ 0 & \pm J & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \pm J & \mp J \\ 0 & \cdots & \cdots & 0 & (\pm 1 \pm (\frac{1}{k} + \frac{l-1}{k}))J \end{pmatrix}.$$

Hence, R is invertible if and only if $\pm 1 \pm \frac{l}{k} \neq 0$. This is clearly the case for all choices of signs if and only if $l \neq \pm k$, completing the proof. \square

For the other direction in Theorem 7.1 we will make use of

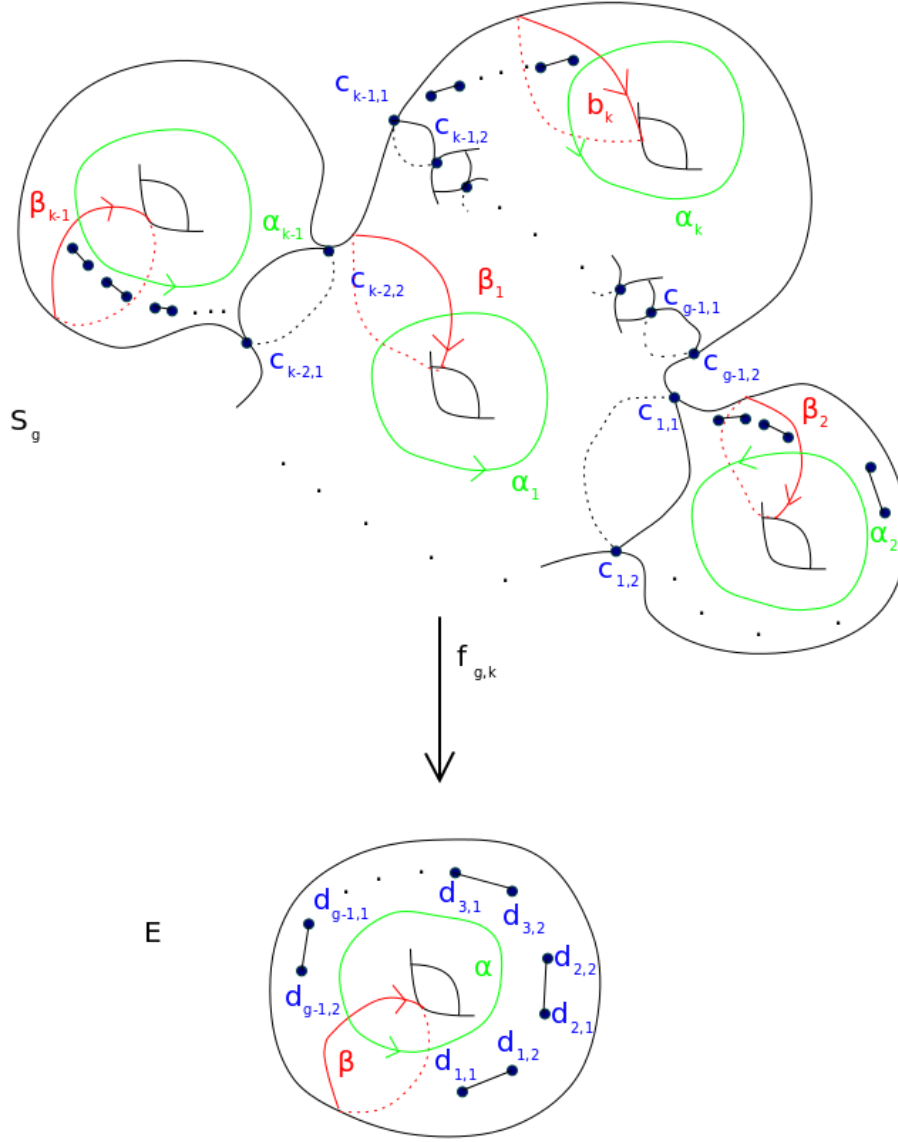


FIGURE 3. The k -fold purely branched covering map $f_{g,k}$ of E in Lemma 7.6

Lemma 7.5. *Let S_g be a closed Riemann surface of genus g and let E be an elliptic curve. Let $f : S_g \rightarrow E$ be a holomorphic k -fold purely branched covering map. Then there exist standard generating sets $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ of $\pi_1 S_g$ and α, β of $\pi_1 E$ such that the induced map on fundamental groups is of the form described in Theorem 7.1(1).*

Proof. Let $B \subset E$ be the finite branching set of f . Since f is purely branched there are generators $\alpha, \beta : [0, 1] \rightarrow E \setminus B$ such that every lift of α and β with respect to the unramified covering $f : S_g \setminus f^{-1}(B) \rightarrow E \setminus B$ is a loop.

We may further assume that the only intersection point of α and β is the point $\alpha(0) = \beta(0)$ and that the intersection number $\iota(\alpha, \beta) = 1$ with respect to the orientation induced by the complex structure on E .

Since $f|_{S_g \setminus f^{-1}(B)}$ is a k -fold unramified covering map, there are precisely k lifts $\alpha_1, \dots, \alpha_k$ of α and β_1, \dots, β_k of β and we may choose them so that $\iota(\alpha_i, \beta_j) = \delta_{ij}$, $\iota(\alpha_i, \alpha_j) = 0$ and $\iota(\beta_i, \beta_j) = 0$.

Then the images of $\alpha_1, \beta_1, \dots, \alpha_k, \beta_k$ in $H_1(S_g, \mathbb{Z})$ forms part of a standard symplectic basis with respect to the symplectic intersection form on $H_1(S_g, \mathbb{Z})$. Extend by $\alpha_{k+1}, \beta_{k+1}, \dots, \alpha_g, \beta_g$ to a standard symplectic basis of $H_1(S_g, \mathbb{Z})$. We claim that with respect to this basis $f_\#$ takes the desired form.

We may assume that $\alpha_{k+1}, \beta_{k+1}, \dots, \alpha_g, \beta_g$ are loops in $S_g \setminus f^{-1}(B)$. Since α_j , $k+1 \leq j \leq g$ forms part of a symplectic basis we have that its intersection number with any of the α_i, β_i with $1 \leq i \leq k$ is zero. Since all of the lifts of α, β are given by $\alpha_1, \beta_1, \dots, \alpha_k, \beta_k$ and the map f is holomorphic, thus orientation preserving, it follows that the intersection numbers $\iota(f \circ \alpha_j, \alpha)$ and $\iota(f \circ \alpha_j, \beta)$ satisfy

$$\begin{aligned}\iota(f \circ \alpha_j, \alpha) &= \sum_{i=1}^k \iota(\alpha_j, \alpha_i) = 0, \\ \iota(f \circ \alpha_j, \beta) &= \sum_{i=1}^k \iota(\alpha_j, \beta_i) = 0.\end{aligned}$$

Nondegeneracy of the symplectic intersection form on $H_1(E, \mathbb{Z}) = \pi_1 E = \langle \alpha, \beta \mid [\alpha, \beta] \rangle$ then implies that $f \circ \alpha_j = 0$ in $\pi_1 E$. Similarly $f \circ \beta_j = 0$ in $\pi_1 E$ for $k+1 \leq j \leq g$.

Since by definition $f \circ \alpha_i = \alpha$ and $f \circ \beta_i = \beta$ for $1 \leq i \leq k$, it follows that with respect to the standard generating sets $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ of $\pi_1 S_g$ and α, β of $\pi_1 E$, the induced map on fundamental groups is indeed

$$\begin{aligned}f_\# : \quad \pi_1 S_g &\rightarrow \pi_1 E \\ \alpha_1, \dots, \alpha_k &\mapsto \alpha \\ \beta_1, \dots, \beta_k &\mapsto \beta \\ \alpha_{k+1}, \dots, \alpha_g &\mapsto 0 \\ \beta_{k+1}, \dots, \beta_g &\mapsto 0.\end{aligned}$$

□

Finally, we prove the existence of a k -fold purely branched holomorphic covering map with Morse type singularities for every $g \geq 2$ and $2 \leq k \leq g$ by giving an explicit construction in analogy to Construction 2 in Section 5.

Proposition 7.6. *For every $g \geq 2$ and every $2 \leq k \leq g$ there is a k -fold purely branched holomorphic covering map $f_{g,k} : S_g \rightarrow E$ with Morse type singularities.*

Proof. Let E be an elliptic curve, let $g \geq 2$, let $1 \leq k \leq g$ and let $d_{1,1}, d_{1,2}, d_{2,1}, \dots, d_{g-1,1}, d_{g-1,2}$ be $2(g-1)$ points in E and $s_1, \dots, s_{g-1} : [0, 1] \rightarrow E$ be simple, pairwise non-intersecting paths with starting point $s_i(0) = d_{i,1}$ and endpoint $s_i(1) = d_{i,2}$ for $i = 1, \dots, g-1$. Take k copies E_0, E_1, \dots, E_{k-1} of E , cut E_0 open along all of the paths s_i , cut E_i open along the path s_i for $1 \leq i \leq k-2$ and cut E_{k-1} open along the paths s_{k-1}, \dots, s_{g-1} . This produces surfaces F_0, \dots, F_{k-1} with boundary.

Gluing the surfaces F_0, \dots, F_{k-1} in the unique way given by identifying opposite edges in the corresponding boundary components F_0 and each of the F_i we obtain a closed surface of genus g together with a continuous k -fold purely branched covering map $f_{g,k} : S_g \rightarrow E$. Choosing the unique complex structure on S_g that makes $f_{g,k}$ holomorphic we do indeed obtain a k -fold

purely branched holomorphic covering map $f_{g,k} : S_g \rightarrow E$ pictured in Figure 3. Looking at this map in local coordinates it is immediate that all singularities are of Morse type. \square

Proof of Theorem 7.1. (2) implies (1) by Lemma 7.4.

Next we prove that (1) implies (2): By Proposition 7.3 and Corollary 6.6 the integer k in (1) is an invariant of the map f_g up to isomorphisms and by Lemma 7.4 it must coincide with the degree of the branched covering map f_g .

The existence of a k -fold purely branched covering map for $2 \leq k \leq g$ now follows from Lemma 7.6. \square

Remark 7.7. Note that Theorem 6.4 and Theorem 7.1 also allow us to distinguish Kähler groups arising from our construction for which not all maps are purely branched, provided that the purely branched maps do not coincide on fundamental groups up to reordering and choosing suitable standard generating sets.

It seems reasonable to us that there is a further generalisation of Proposition 7.3 to branched covering maps which are not purely branched. A suitable generalisation would allow us to classify all Kähler groups that can arise using our construction up to isomorphism. We are planning to address this question in future work. In particular, the purely branched maps should correspond precisely to the branched covering maps inducing maps of the form $(I \cdots I 0 \cdots 0)$ on homology which would allow us to remove the assumption that f_g is purely branched in Theorem 7.1.

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MATHEMATICAL INSTITUTE, ANDREW WILES BUILDING, UNIVERSITY OF OXFORD, OXFORD OX2 6GG, UK
E-mail address: llosaisenric@maths.ox.ac.uk